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CLOSING	

Global Exponential Periodicity for Discrete Hopfield Neural Networks with Delays and Impulses

Valéry Covachev, Zlatinka Covacheva,
Haydar Akça, Sannay Mohamad

1 Introduction

In the present paper we introduce the discrete counterpart of a class of Hopfield neural networks with periodic integral impulsive conditions and finite distributed delays. We apply the continuation theorem of coincidence degree theory [3] to obtain a sufficient condition for the existence of a periodic solution of the discrete system considered. By introducing an appropriate Lyapunov functional we derive a sufficient condition for the uniqueness and global exponential stability of the periodic solution.

2 Statement of the problem. Main results

We consider a class of Hopfield neural networks with periodic integral impulsive conditions and finite distributed delays, which are formulated in the form of a system of impulsive delay differential equations

$$\begin{aligned} \frac{dx_i}{dt} &= -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t) f_j\left(\int_0^{\omega} g_{ij}(s)x_j(t-s)\,ds\right) + I_i(t), \\ & \quad t \neq t_k, \\ \Delta x_i(t_k) &\equiv x_i(t_k+0) - x_i(t_k) \\ &= -\gamma_{ik}x_i(t_k) + \sum_{j=1}^{m} B_{ijk}\Phi_j\left(\int_0^{\omega} c_{ij}(s)x_j(t_k-s)\,ds\right) + \alpha_{ik}, \\ & \quad i = \overline{1,m}, \quad k \in \mathbb{Z}, \end{aligned} \tag{1}$$

where m is the number of neurons in the network, $x_i(t)$ is the state of the i-th neuron at time t, $a_i(t) > 0$ is the rate at which the i-th neuron resets its state when isolated from the system, $b_{ij}(t)$ is the synaptic connection weight from the j-th neuron to the i-th one, $f_j(\cdot)$ are signal transmission functions of the j-th neuron, ω is the maximum transmission delay from one neuron to another, $g_{ij}(\cdot)$ and $c_{ij}(\cdot)$ are nonnegative delay kernels, $I_i(t)$ is the external input to the i-th neuron, t_k ($k \in \mathbb{Z}$) are the instants of impulse effect which form a strictly increasing sequence, γ_{ik} ($i = \overline{1,m},\ k \in \mathbb{Z}$) are positive constants.

We assume that the above system (1) satisfies the following periodicity conditions: $a_i(t)$, $b_{ij}(t)$, $I_i(t)$ are ω-periodic in t; $t_{k+p} = t_k + \omega$, $\gamma_{i,k+p} = \gamma_{ik}$, $B_{ij,k+p} = B_{ijk}$, $\alpha_{i,k+p} = \alpha_{ik}$.

As in [5, 1] we formulate the discrete counterpart of system (1). For a positive integer N we choose the discretization step $h = \omega/N$. For the moment we assume N so large that $h < \min\limits_{k=\overline{1,p}} (t_{k+1} - t_k)$. Then each interval $[nh, (n+1)h]$ contains at most one instant of impulse effect t_k.

For convenience we denote $n = [t/h]$, the greatest integer in t/h, and $n_k = [t_k/h]$. Clearly, we will have $n_{k+p} = n_k + N$ for all $k \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$, $n \neq n_k$. This means that the interval $[nh, (n+1)h]$ contains no instant of impulse effect t_k. First we approximate the integral term in (1) by a sum and then we approximate the differential equation (1) on the interval $[nh, (n+1)h]$ by

$$\frac{dx_i}{dt} + a_i(nh)x_i(t) = I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh) f_j \left(\sum_{\ell=1}^{N} g_{ij}(\ell h) x_j\left((n-\ell)h\right) \varphi(h) \right).$$

We multiply both sides of this equation by $\exp\left(a_i(nh)t\right)$ and integrate over the interval $[nh, (n+1)h]$. Thus we obtain

$$\begin{aligned} x_i\left((n+1)h\right) - x_i(nh) &= -\left(1 - e^{-a_i(nh)h}\right) x_i(nh) \\ + \quad \frac{1 - e^{-a_i(nh)h}}{a_i(nh)} &\left\{ I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh) f_j \left(\sum_{\ell=1}^{N} g_{ij}(\ell h) x_j\left((n-\ell)h\right) \varphi(h) \right) \right\}. \end{aligned} \tag{2}$$

Henceforth by abuse of notation we write $x_i(n) = x_i(nh)$ and define $\Delta x_i(n) = x_i(n+1) - x_i(n)$ $(i = \overline{1,m},\ n \in \mathbb{Z})$.

Next, for $n = n_k$ the interval $[nh, (n+1)h]$ contains the instant of impulse effect t_k. On this interval we approximate the impulse condition in (1) by

$$\begin{aligned} \Delta x_i(n_k) \quad &= \quad -\gamma_{ik} x_i(n_k) + \alpha_{ik} + \sum_{j=1}^{m} B_{ijk} \Phi_j \left(\sum_{\ell=1}^{N} c_{ij}(\ell) x_j(n_k - \ell) \right), \\ & i = \overline{1,m}, \quad k \in \mathbb{Z}. \end{aligned} \tag{3}$$

Introducing some notations, we can write the difference system (2), (3) in operator form as

$$\Delta x = Hx, \tag{4}$$

where

$$\begin{aligned} (Hx)_i(n) &= -A_i(n)x_i(n) + I_i(n) \\ + &\begin{cases} \sum\limits_{j=1}^{m} b_{ij}(n) f_j \left(\sum\limits_{\ell=1}^{N} g_{ij}(\ell) x_j(n-\ell) \right), & n \neq n_k, \\ \sum\limits_{j=1}^{m} B_{ijk} \Phi_j \left(\sum\limits_{\ell=1}^{N} c_{ij}(\ell) x_j(n_k - \ell) \right), & n = n_k. \end{cases} \end{aligned}$$

In order to formulate our assumptions, we need some more notation:

$$I_N = \{0,\, 1,\, \ldots,\, N-1\},$$

$$\underline{A}_i = \min_{n \in I_N} A_i(n), \quad \overline{A}_i = \sum_{n=0}^{N-1} A_i(n), \qquad i = \overline{1,m}.$$

Now we introduce the following conditions:

H1. $A_i(n+N) = A_i(n)$, $I_i(n+N) = I_i(n)$ for $i = \overline{1,m}$, $n \in \mathbb{Z}$;
$n_k \in \mathbb{Z}$ for all $k \in \mathbb{Z}$ and $n_{k+p} = n_k + N$; $b_{ij}(n+N) = b_{ij}(n)$ $(n \neq n_k)$, $B_{ij,k+p} = B_{ijk}$ $(k \in \mathbb{Z})$ for $i,j = \overline{1,m}$.

H2. $\underline{A}_i > 0$, $\overline{A}_i < 1$ for $i = \overline{1,m}$.

H3. The functions $f_j(\cdot)$, $\Phi_j(\cdot)$ $(j = \overline{1,m})$ are bounded on $\mathbb{R}$ and there exist positive constants M_j and L_j such that

$$|f_j(x) - f_j(y)| \le M_j|x-y|, \quad |\Phi_j(x) - \Phi_j(y)| \le L_j|x-y|$$

for all $x, y \in \mathbb{R}$.

H4. $g_{ij}(\ell) \ge 0$, $c_{ij}(\ell) \ge 0$ for $i,j = \overline{1,m}$, $\ell = \overline{1,N}$.

We again introduce some notation:

$$\overline{I}_i = \max_{n \in I_N} |I_i(n)|, \qquad i = \overline{1,m},$$

$$\overline{b}_{ij} = \sup_{n \neq n_k} |b_{ij}(n)|, \quad \overline{B}_{ij} = \max_{k=\overline{1,p}} |B_{ijk}|, \qquad i,j = \overline{1,m}.$$

For an N-periodic sequence $v(n)$ we denote $\tilde{v} = \frac{1}{N}\sum\limits_{n=0}^{N-1} v(n)$; for $i = \overline{1,m}$

$$\rho_i = \overline{I}_i + \frac{1}{N}\sum_{j=1}^{m}\left[(N-p)\overline{b}_{ij}|f_j(0)| + p\overline{B}_{ij}|\Phi_j(0)|\right].$$

Next we denote

$$\mathcal{M}_j = \max\{L_j,\ M_j\}, \qquad\qquad j = \overline{1,m},$$

$$G_{ij} = \sum_{\ell=1}^{N} g_{ij}(\ell), \quad C_{ij} = \sum_{\ell=1}^{N} c_{ij}(\ell), \qquad i,j = \overline{1,m},$$

$$\mathcal{B}_{ij} = \max\{\overline{b}_{ij},\ \overline{B}_{ij}\}, \quad \mathcal{G}_{ij} = \max\{G_{ij},\ C_{ij}\}, \qquad i,j = \overline{1,m}.$$

We introduce the $m \times m$ matrices

$$A = \operatorname{diag}\left(\frac{\underline{A}_i\left(1-\overline{A}_i\right)}{1+N\underline{A}_i},\ i = \overline{1,m}\right), \qquad B = \left(\mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}\right)_{i,j=1}^{m}.$$

Then we introduce the conditions

H5. $\min_{i=\overline{1,m}} \left(\tilde{A}_i - \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji} \right) > 0.$

H6. $\underline{A}_i > \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji}$ for $i = \overline{1,m}$.

H7. The matrix $A - B$ is an M-matrix [2, 4].

Clearly, condition **H6** implies **H5** but the converse is not true. Condition **H7** means that the matrix $A - B$ is nonsingular and its inverse has positive entries only.
Our results are formulated as follows:

Theorem 1 *Suppose that conditions* **H1**–**H5***,* **H7** *hold. Then the equation* (4) *has at least one* N*-periodic solution.*

Theorem 1 is proved using Mawhin's continuation theorem [3, p. 40].

Theorem 2 *Suppose that conditions* **H1**–**H4***,* **H6***,* **H7** *hold. Then the* N*-periodic solution of* (4) *is unique and globally exponentially stable.*

In fact, let us suppose that $x^*(n) = (x_1^*(n), x_2^*(n), \ldots, x_m^*(n))^T$ is an N-periodic solution of equation (4), and $x(n) = (x_1(n), x_2(n), \ldots, x_m(n))^T$ is any solution of (4) for $n \geq 0$, defined at least for $n \geq -N$. Then by introducing an appropriate Lyapunov functional we derive the estimate

$$\sum_{i=1}^{m} |x_i(n) - x_i^*(n)| \leq C\lambda^{-n} \sum_{i=1}^{m} \max_{s \in I_{-N}} |x_i(s) - x_i^*(s)|, \quad n \in \mathbb{Z}_0^+,$$

where $C > 0$ and $\lambda > 1$.

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